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Scalar and tensor spherical harmonics expansion of the velocity correlation in homogeneous anisotropic turbulence

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The representation theory of the rotation group is applied to construct a series expansion of the correlation tensor in homogeneous anisotropic turbulence. The resolution of angular dependence is the main analytical difficulty posed by anisotropic turbulence; representation theory parametrizes this dependence by a tensor analog of the standard spherical harmonics expansion of a scalar. The series expansion is formulated in terms of explicitly constructed tensor bases with scalar coefficients determined by angular moments of the correlation tensor.

1 Introduction

The velocity correlation tensor in homogeneous isotropic turbulence can be characterized by a scalar function of a scalar argument, for example, by the energy spectrum as a function of wavenumber. If the assumption of isotropy is dropped, such a simple general representation is no longer possible. The simplest possible case of anisotropic turbulence is axial symmetry with reflection invariance in the plane perpendicular to the symmetry axis. The earliest investigations [1, 2] of this problem found that a general expression for the correlation tensor requires two fixed tensors and two scalar coefficients that are both functions of a *wavevector* argument. The substantial increase in complexity compared to isotropy is obvious.

Only more recently has the correlation tensor in general anisotropic homogeneous turbulence been analyzed by applying the 'SO(3) decomposition' [3, 4], a generalization of spherical harmonics expansions to tensors of any rank. The outcome of this analysis is a decomposition of the correlation tensor into components that transform according to different *irreducible representations* of the

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rotation group indexed by an integral *spin* [5]. Needless to say, tensor analogs of the scalar spherical harmonics expansions arise in many problems besides turbulence: Ref. [6] is only one example.

In this paper, we take these constructions one step further, and show how to write the tensor analogs of spherical harmonics, or *tensor spherical harmonics*, using a basis of fixed anisotropic tensors. This result generalizes the representation of axial symmetry by one anisotropic tensor [1, 2] to arbitrary anisotropy. The analysis is based on a preliminary decomposition of the correlation tensor into *directional* and *polarization* components following Cambon and Rubinstein [7]. As directional and polarization anisotropy belong to rotation invariant vector spaces, their introduction into the SO(3) analysis is geometrically natural.

Directional anisotropy is 'tensorially isotropic' since, as in axial symmetry, anisotropy enters its description only through a scalar function of a wavevector argument. The SO(3) decomposition for directional anisotropy then reduces to a standard spherical harmonics expansion. Polarization anisotropy is the trace-free, or deviatoric, part of the correlation tensor. It requires a considerably more complicated description than directional anisotropy. We use results from Arad et al. [3] to construct polarization tensors of all spins.

Whereas directional anisotropy proves to be restricted to even spins, polarization anisotropy admits both even and odd spins; the latter can arise in applications with frame rotation or mean strain fields with an antisymmetric part. Polarization tensors of even spin are shown to be linear combinations of five tensors, with polynomial coefficients related to ordinary spherical harmonics. An elementary geometric argument shows that a general polarization tensor is determined by only two scalar coefficients [8] (see also [6]); we will demonstrate that this reduction requires replacing the polynomial coefficients of the SO(3) expansion by rational functions and discuss the implications of this step.

Polarization tensors of odd spin are expressed similarly in terms of a second set of five basis tensors. The solenoidal condition prohibits the existence of spin one polarization tensors, so that the minimum possible spin is three. We discuss a connection between the *stropholysis* tensor defined by Kassinos et al. [9] and spin three polarization.

The developments discussed so far apply to non-helical turbulence; a brief treatment of the helical case is also included for completeness.

Although this paper is devoted to the essentially formal problem of constructing a series expansion without any specific application, it is motivated by the implications of the SO(3) analysis for turbulence modeling. The difficulty posed by anisotropy is that the angle dependence must be described by functions of up to three independent variables. What the SO(3) analysis does is to parametrize this dependence in terms of fixed scalar and tensor functions multiplied by scalar amplitudes that depend on wavenumber alone. Thus, the work of Clark and Zemach [10] used SO(3) decompositions to overcome the limitations of a spectral closure based on 'angle-averaged' variables proposed by Besnard et al. [11], in which the details of the anisotropic dynamics are necessarily suppressed. The SO(3) analysis replaces angle averages by more general angular moments thereby permitting complete resolution of the angle-dependent dynamics.

The reduction to one independent variable compensates for the introduction of many unknown amplitudes; moreover, in turbulence, it is anticipated that nonlinear processes of 'return to isotropy' may permit a usable description with a relatively small number of angular harmonics. The replacement of general angle dependence by moments can also link two-point closure theories to the single-point quantities used in turbulence models. This connection will be the subject of future research.

The paper is organized as follows. Section 2 introduces the basic directional-polarization decomposition. Section 3 uses representation theory to construct bases for expansion of directional and polarization anisotropy. Section 4 treats the antisymmetric part of the correlation tensor. Section 5 contains a summary and conclusions.

2 The directional-polarization decomposition

Following Ref. [7], index and index-free notation will both be used: vectors are denoted by a_i or $\boldsymbol{a}$, tensors by A_{ij} or A, tensor products by $a_i b_j$ or $\boldsymbol{ab}$, and contractions of tensors by $A_{mn}B_{mn}$ or $\mathsf{A} : \mathsf{B}$. Vector and tensor indices are denoted by the usual Latin letters $i, j, \cdots$ and are governed by the summation convention; Greek letters $\nu, \mu, \cdots$ are used for enumerations and are not governed by the summation convention. Finally, the same symbol may denote different quantities that are distinguished by their arguments and/or indices.

The Fourier wavevector representation will be used, in which the correlation tensor is the function $U_{ij}(\boldsymbol{k})$ of wavevector $\boldsymbol{k}$. Its definition implies $U_{ij}(\boldsymbol{k}) = U_{ji}(-\boldsymbol{k})$, (for example, Monin and Yaglom [12] Eq. (11.56)). Consequently,

$$\text{if } U_{ij}(\boldsymbol{k}) = U_{ji}(\boldsymbol{k}) \text{ then } U_{ij}(-\boldsymbol{k}) = U_{ij}(\boldsymbol{k}) \tag{1}$$

$$\text{if } U_{ij}(\boldsymbol{k}) = -U_{ji}(\boldsymbol{k}) \text{ then } U_{ij}(-\boldsymbol{k}) = -U_{ij}(\boldsymbol{k}) \tag{2}$$

An immediate consequence is that, in order that the inverse Fourier transform of the correlation tensor be real, the index symmetric part of the correlation tensor must be real, and the index antisymmetric part, corresponding to helicity, must be purely imaginary; these issues are discussed more completely by Cambon and Jacquin [13].

Incompressibility requires $U_{ij}(\boldsymbol{k})$ to be solenoidal:

$$k_i U_{ij}(\boldsymbol{k}) = k_j U_{ij}(\boldsymbol{k}) = 0. \tag{3}$$

For isotropy, the correlation has the form

$$U_{ij}(\boldsymbol{k}) = U(k) P_{ij}(\boldsymbol{k}) \tag{4}$$

where $P_{ij}(\boldsymbol{k})$ is the transverse projection matrix

$$P_{ij}(\boldsymbol{k}) = \delta_{ij} - k^{-2} k_i k_j \tag{5}$$

Ref. [7] formulates the directional-polarization decomposition as follows: any solenoidal $U_{ij}(\boldsymbol{k})$ can be written

$$U_{ij}(\boldsymbol{k}) = U_{ij}^{dir}(\boldsymbol{k}) + U_{ij}^{pol}(\boldsymbol{k}) \tag{6}$$

where $U_{ij}^{dir}(\boldsymbol{k})$ is the geometric projection of $U_{ij}(\boldsymbol{k})$ onto $P_{ij}(\boldsymbol{k})$,

$$U_{ij}^{dir}(\boldsymbol{k}) = \tfrac{1}{2}(\mathsf{U}(\boldsymbol{k}) : \mathsf{P}(\boldsymbol{k})) P_{ij}(\boldsymbol{k}) \tag{7}$$

(the factor $\frac{1}{2}$ appears because $\mathsf{P} : \mathsf{P} = \mathrm{tr}\ \mathsf{P} = 2$) and U_{ij}^{pol} is the remainder

$$U_{ij}^{pol}(\boldsymbol{k}) = U_{ij}(\boldsymbol{k}) - U_{ij}^{dir}(\boldsymbol{k}) \tag{8}$$

Then $U_{ij}^{pol}(\boldsymbol{k})$ is both trace-free and solenoidal. We will call any such tensor a *polarization* tensor. It is convenient to write Eq. (7) as

$$U_{ij}^{dir}(\boldsymbol{k}) = U^{dir}(\boldsymbol{k}) P_{ij}(\boldsymbol{k}) \tag{9}$$

where

$$U^{dir}(\boldsymbol{k}) = \tfrac{1}{2}\mathsf{U}(\boldsymbol{k}) : \mathsf{P}(\boldsymbol{k}) = \tfrac{1}{2}\mathrm{tr}\ \mathsf{U}(\boldsymbol{k}) \tag{10}$$

Thus, directional anisotropy is characterized by a scalar function of a wavevector argument; since directional anisotropy is proportional to the isotropic tensor $P_{ij}(\boldsymbol{k})$, it may be said to be tensorially isotropic. As in the trace-deviator decomposition, directional and polarization anisotropy transform under rotation into components of the same type.

Our next step is to decompose directional and polarization anisotropy into irreducible representations of SO(3) following Refs. [3, 5].

3 SO(3) decompositions

3.1 *Directional anisotropy*

The SO(3) decomposition of $U^{dir}(\boldsymbol{k})$ is the series expansion

$$U^{dir}(\boldsymbol{k}) = U(k) + H^{dir}_{mn}(k)k^{-2}k_m k_n + H^{dir}_{mnrs}(k)k^{-4}k_m k_n k_r k_s + \cdots \tag{11}$$

where $U(k)$ is the isotropic contribution to $U^{dir}(\boldsymbol{k})$, and the anisotropic contribution is described by polynomials in the components of $\boldsymbol{k}$ with coefficient tensors H^{dir} that are functions only of wavenumber k. Since $P_{ij}(\boldsymbol{k}) = P_{ji}(\boldsymbol{k})$ obeys index symmetry, Eq. (1) requires $U^{dir}(\boldsymbol{k}) = U^{dir}(-\boldsymbol{k})$ and restricts the terms in Eq. (11) to even degree polynomials.

The coefficient tensors $H^{dir}_{mn}(k)$, $H^{dir}_{mnrs}(k)$, $\cdots$ can be assumed symmetric under any interchange of indices. They are also trace-free, in the extended sense that contraction of any two indices vanishes identically. There are $2\nu+1$ linearly independent tensors with this property for each degree ν. Then if the argument of $H^{dir}_{m_1,\cdots m_\nu}(k)$ is fixed, the polynomial $H^{dir}_{m_1,\cdots m_\nu}k_{m_1}\cdots k_{m_\nu}$ is harmonic and homogeneous of degree ν [5].

Such polynomials, or equivalently, their corresponding coefficient tensors, define the irreducible representations [3, 5] of the rotation group. They form a vector space of dimension $2\nu+1$ that will be denoted by H_ν. In applications to physics, ν is called the *spin*, and although this term has no particular relevance in the context of turbulence, we will use this standard terminology; here, the spin is the rank of the coefficient tensor or the degree of the corresponding homogeneous polynomial. Note that spin zero corresponds to the isotropic part $U(k)$.

The terms in Eq. (11) are defined as follows: $U(k)$ is simply the spherical average

$$U(k) = \frac{1}{4\pi k^2}\oint dS(\boldsymbol{k})U^{dir}(\boldsymbol{k}) \tag{12}$$

where the integration symbol denotes integration over a sphere of radius k. Practical difficulties notwithstanding, the importance of averaging over all directions in extracting $U(k)$ from experimental data was demonstrated by Kurien and Sreenivasan [14].

The first two H^{dir} tensors are defined by

$$\tfrac{2!}{5\cdot 3}H^{dir}_{ij}(k) = \frac{1}{4\pi k^2}\oint dS(\boldsymbol{k})\ U^{dir}(\boldsymbol{k})k^{-2}T_{ij}(\boldsymbol{k}) \tag{13}$$

$$\tfrac{4!}{9\cdot 7\cdot 5\cdot 3}H^{dir}_{ijmn}(k) = \frac{1}{4\pi k^2}\oint dS(\boldsymbol{k})\ U^{dir}(\boldsymbol{k})k^{-4}T_{ijmn}(\boldsymbol{k}) \tag{14}$$

where

$$T_{ij}(\boldsymbol{k}) = k_i k_j - \tfrac{1}{3}k^2\delta_{ij} \tag{15}$$
$$T_{ijmn}(\boldsymbol{k}) = k_i k_j k_m k_n - \tfrac{1}{7}k^2(\delta_{ij}k_m k_n + \text{perm.}) + \tfrac{1}{35}k^4(\delta_{ij}\delta_{mn} + \text{perm.}) \tag{16}$$

and "perm." denotes summation over all index permutations that lead to a distinct result (thus, there are 6 terms in the second contribution to T_{ijmn} in Eq. (16) and 3 in the third). The tensors T_{ij} and T_{ijmn} are the unique trace-free isotropic tensors containing $k_i k_j$ and $k_i k_j k_m k_n$ respectively. These objects and their integrals are ubiquitous in applications (for example, Waldmann [15]). They are readily generalized to higher spins and are used to construct the corresponding H^{dir} tensors in Eq. (11).

The expansion Eq. (11) has the property that

$$\oint dS(\boldsymbol{k}) k_{n_1}\cdots k_{n_\nu} U^{dir}(\boldsymbol{k}) = \oint dS(\boldsymbol{k}) k_{n_1}\cdots k_{n_\nu} \times$$
$$\left(U(k) + H^{dir}_{m_1 m_2}(k)k^{-2}k_{m_1}k_{m_2} + \cdots + H^{dir}_{m_1\cdots m_\nu}(k)k^{-\nu}k_{m_1}\cdots k_{m_\nu}\right) \tag{17}$$

so that any moment of U^{dir} of order ν is determined by the first ν terms of the series. The expansion Eq. (11) can in fact be derived from this property.

3.2 *Basis of spherical harmonics*

The SO(3) analysis decomposes the space of functions of wavevector $\boldsymbol{k}$ into the rotation-invariant subspaces H_ν of dimension $2\nu + 1$. Constructing a basis for these subspaces can be useful for analysis. There is no unique construction, but a standard one uses the invariance of the H_ν under the group SO(2) of two-dimensional rotations about an axis $\boldsymbol{a}$. Since SO(2) is a subgroup of SO(3), the irreducible representations of SO(3) on the spaces H_ν break up into irreducible representations of SO(2). This happens in a standard way [5] and leads to the familiar Legendre polynomials and spherical harmonics.

We begin with the identity representation [5] of SO(2) on axisymmetric harmonic polynomials. There is only one such polynomial of each spin; therefore, for axial symmetry, the expansion Eq. (11) simplifies to

$$U^{dir}(\boldsymbol{k}) = \sum_{\nu\geq 0,\ \text{even}} A_\nu(k)k^{-\nu}Y^\nu(\boldsymbol{k}) \tag{18}$$

where the $Y^\nu(\boldsymbol{k})$ are Legendre polynomials; for later calculations, they must be expressed as homogeneous polynomials in the components of $\boldsymbol{k}$; thus, if unit vector $\boldsymbol{a}$ is the symmetry axis,

$$\begin{aligned}
Y^0(\boldsymbol{k}) &= 1 \\
Y^2(\boldsymbol{k}) &= k^2 - 3(\boldsymbol{a}\cdot\boldsymbol{k})^2 \\
Y^4(\boldsymbol{k}) &= 3k^4 - 30k^2(\boldsymbol{a}\cdot\boldsymbol{k})^2 + 35(\boldsymbol{a}\cdot\boldsymbol{k})^4
\end{aligned} \tag{19}$$

Here and in what follows, in place of the standard normalizations, such polynomials are written with relatively prime integer coefficients.

Let $\boldsymbol{k} = (k_x, k_y, k_z)$ with $k_z = \boldsymbol{a}\cdot\boldsymbol{k}$. For each spin ν and μ such that $0 \le \mu \le \nu$, define $Y^{\nu,\pm\mu}(\boldsymbol{k})$ by

$$Y^{\nu,\mu}(\boldsymbol{k}) + iY^{\nu,-\mu}(\boldsymbol{k}) = c_{\nu,\mu}(k_x + ik_y)^\mu \frac{\partial^\mu}{\partial k_z^\mu} Y^\nu(\boldsymbol{k}) \tag{20}$$

where the derivative $\partial/\partial k_z$ is evaluated treating k and k_z as independent variables in Eq. (19) and its generalizations to higher spins. Up to normalization, the result is the usual spherical harmonics written as homogeneous polynomials. Here and subsequently, $c_{\nu,\mu}$ denotes constants chosen to simplify the final expressions. They are not necessarily the same constants each time $c_{\nu,\mu}$ appears. For example, for spin two, they can be chosen so that

$$\begin{aligned}
&Y^{2,0}(\boldsymbol{k}) = Y^2(\boldsymbol{k}) = k^2 - 3k_z^2 \\
&Y^{2,1}(\boldsymbol{k}) = k_x k_z \quad Y^{2,-1}(\boldsymbol{k}) = k_y k_z \\
&Y^{2,2}(\boldsymbol{k}) = k_x^2 - k_y^2 \quad Y^{2,-2}(\boldsymbol{k}) = 2k_x k_y
\end{aligned} \tag{21}$$

Using spherical coordinates $k_x = k\sin\theta\cos\phi$, $k_y = k\sin\theta\sin\phi$, $k_z = k\cos\theta$, and complex exponentials $k_x + ik_y = k\sin\theta\ e^{i\phi}$,

$$\begin{aligned}
Y^{2,0}(\boldsymbol{k}) &= Y^2(\boldsymbol{k}) = k^2(1 - 3\cos^2\theta) \\
Y^{2,1}(\boldsymbol{k}) + iY^{2,-1}(\boldsymbol{k}) &= k^2\cos\theta\ \sin\theta\ \exp(i\phi) \\
Y^{2,2}(\boldsymbol{k}) + iY^{2,-2}(\boldsymbol{k}) &= k^2\sin^2\theta\ \exp(2i\phi)
\end{aligned} \tag{22}$$

The complex exponentials $\exp(im\phi)$ define the irreducible representations of SO(2) [5], so that their appearance in Eq. (22) and its generalizations to higher spins is natural.

The general expansion Eq. (11) can be written in this basis as

$$U^{dir}(\boldsymbol{k}) = \sum_{\nu\ \text{even}, -\nu\le\mu\le\nu} A_{\nu,\mu}(k) k^{-\nu} Y^{\nu,\mu}(\boldsymbol{k}) \tag{23}$$

Comparing with the terms in Eq. (11),

$$H^{dir}_{mn}(k)k^{-2}k_m k_n = \sum_{-2\leq\mu\leq 2} A_{2,\mu}(k)k^{-2}Y^{2,\mu}(\boldsymbol{k}) \tag{24}$$

$$H^{dir}_{mnrs}(k)k^{-4}k_m k_n k_r k_s = \sum_{-4\leq\mu\leq 4} A_{4,\mu}(k)k^{-4}Y^{4,\mu}(\boldsymbol{k}) \tag{25}$$

As required, there are 5 and 9 terms in the basis for spin two (Eq. (24)) and spin four (Eq. (25)) respectively. We emphasize that SO(3) analysis provides no natural a basis for the spaces H_ν and that the standard spherical harmonics basis developed here requires choosing an arbitrary polar axis $\boldsymbol{a}$. The basis functions $Y^{\nu,\mu}$ depend on this choice, but this dependence is quantified by standard formulas of representation theory [5].

3.3 *Polarization anisotropy: even spins*

The tensor $U^{pol}_{ij}(\boldsymbol{k})$ must also be expanded appropriately, but this is a much more complicated matter. Beginning with the case of index symmetry Eq. (1), we will obtain an expansion in tensors with polynomial components of increasing degree analogous to Eq. (11),

$$U^{pol}_{ij}(\boldsymbol{k}) = H^{pol}_{ijmn}(k)k^{-2}k_m k_n + H^{pol}_{ijmnrs}(k)k^{-4}k_m k_n k_r k_s + \cdots \tag{26}$$

where, as in Eq. (11), the expansion is restricted to polynomials of even degree by Eq. (1). Since isotropy is included in the directional expansion, the polarization expansion starts at spin two.

From the viewpoint of SO(3), the problem posed by polarization anisotropy is the decomposition into irreducible representations of a representation of the rotation group on a vector space of tensors with polynomial entries. To construct this decomposition, we will use the differential operator formalism of Ref. [3], which generates the required tensors by operating on scalar functions with rotation-invariant matrices of differential operators. The formulations in Refs. [4, 6] differ in details but are essentially equivalent.

Consider the linear combination of the differential operators B_1, B_5, B_7, and B_9 of Ref. [3] Eq. (10):

$$\mathcal{L}^\nu_{ij} = k^2\partial_i\partial_j - (\nu-1)(k_i\partial_j + k_j\partial_i) + \tfrac{1}{2}\nu(\nu-1)(\delta_{ij} + k_i k_j k^{-2}) \tag{27}$$

where, to lighten the notation, ∂_i is written for $\partial/\partial k_i$; compare also L'vov et al. [16] for the use of Fourier variables. This operator has the property that $\mathcal{L}^\nu_{ij}[\Phi_\nu(\boldsymbol{k})]$ is solenoidal for any $\Phi_\nu(\boldsymbol{k})$ which is homogeneous of degree ν.

Moreover, if Φ_ν is also harmonic, then

$$\delta_{ij}\mathcal{L}^{\nu}_{ij}[\Phi_\nu(\boldsymbol{k})] = 0 \tag{28}$$

so that $\mathcal{L}^{\nu}_{ij}[\Phi_\nu(\boldsymbol{k})]$ is both solenoidal and trace-free, hence a polarization tensor. These properties depend on the Euler homogeneity relation $k_i\partial_i\Phi_\nu = \nu\Phi_\nu$. A similar operator that generates solenoidal tensors appears in Ref. [16], but that operator does not generate trace-free tensors.

In order to obtain tensors with even order polynomial components, we consider even values of ν only, and consider odd values subsequently. The tensor analog of the expansion Eq. (17) is

$$U^{pol}_{ij}(\boldsymbol{k}) = \mathcal{L}^2_{ij}[p_2(\boldsymbol{k})] + \mathcal{L}^4_{ij}[p_4(\boldsymbol{k})] + \cdots \tag{29}$$

where $p_n(\boldsymbol{k})$ is a homogeneous harmonic polynomial in $\boldsymbol{k}$ with coefficients that can depend on k. It is understood that the $\mathcal{L}^{\nu}_{ij}$ act on the harmonic polynomials obtained by holding the wavenumber argument of the coefficient tensors constant: recall that the characteristic properties of $\mathcal{L}^{\nu}_{ij}$ require that it act on harmonic polynomials.

Writing, for example, $p_2(\boldsymbol{k}) = a_{mn}(k)k_m k_n$, we obtain

$$\mathcal{L}^2_{ij}[p_2(\boldsymbol{k})] = 2[k^2 a_{ij} - (k_i k_n a_{jn} + k_j k_n a_{in}) + \tfrac{1}{2}(\delta_{ij} + k_i k_j k^{-2})a_{mn}k_m k_n] \tag{30}$$

and thus, equating this result to the spin two contribution to Eq. (26),

$$H^{pol}_{ijmn} = 2[a_{ij}\delta_{mn} - (\delta_{im}a_{jn} + \delta_{jm}a_{in}) + \tfrac{1}{2}(\delta_{ij} + k_i k_j k^{-2})a_{mn}] \tag{31}$$

Analogous results for the higher spins follow similarly. We remark that the right side of Eq. (30) can also be written following Ref. [7], using a solenoidal projection operator, as

$$k^2(P_{im}(\boldsymbol{k})P_{jn}(\boldsymbol{k}) + P_{jm}(\boldsymbol{k})P_{in}(\boldsymbol{k}) - P_{ij}(\boldsymbol{k})P_{mn}(\boldsymbol{k}))a_{mn} \tag{32}$$

Retaining only spin two directional and polarization anisotropy from Eqs. (11) and (26) gives a model for the correlation tensor

$$U_{ij}(\boldsymbol{k}) = (\tfrac{1}{2}U(k) + H^{dir}_{mn}(k)k^{-2}k_m k_n)P_{ij}(\boldsymbol{k}) + H^{pol}_{ijmn}(k)k^{-2}k_m k_n \tag{33}$$

discussed in Ref. [7] and used recently as the basis of a spectral model by Mons et al. [17]. The present work shows how to generalize this model to higher even spin directional and polarization anisotropy, and to odd spin polarization anisotropy.

3.4 *Basis of even spin tensor spherical harmonics*

As in the analysis of directional anisotropy, a basis for the tensors of spin ν can be introduced. In this problem, it is helpful to treat axial symmetry first in more detail.

It follows from Ref. [3] that there is essentially (that is, up to a multiple of a function of k) only one axisymmetric polarization tensor for each spin ν, and it is obtained by applying $\mathcal{L}^{\nu}_{ij}$ to $Y^{\nu}(\boldsymbol{k})$. An elementary calculation shows that

$$\mathcal{L}^{\nu}_{ij}[Y^{\nu}(\boldsymbol{k})] = \frac{\partial^2 Y^{\nu}(\boldsymbol{k})}{\partial k_z^2} S^{pol}_{ij}(\boldsymbol{k}) \tag{34}$$

where

$$S^{pol}_{ij}(\boldsymbol{k}) = k^2 a_i a_j - (\boldsymbol{a}\cdot\boldsymbol{k})(k_i a_j + k_j a_i) + \tfrac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{k})^2[\delta_{ij} + k^{-2} k_i k_j] - \tfrac{1}{2}k^2 P_{ij}(\boldsymbol{k}) \tag{35}$$

is a polarization tensor. The important fact, special to axial symmetry, is that it is the same tensor for all ν. Defining

$$Z^{\nu}(\boldsymbol{k}) = c_{\nu} \frac{\partial^2 Y^{\nu}(\boldsymbol{k})}{\partial k_z^2} \tag{36}$$

the first few of these functions are

$$\begin{aligned} Z^0(\boldsymbol{k}) &= 1 \\ Z^2(\boldsymbol{k}) &= 7(\boldsymbol{a}\cdot\boldsymbol{k})^2 - k^2 \\ Z^4(\boldsymbol{k}) &= 33(\boldsymbol{a}\cdot\boldsymbol{k})^4 - 18k^2(\boldsymbol{a}\cdot\boldsymbol{k})^2 + k^4 \end{aligned} \tag{37}$$

where the c_{ν} have been chosen to make the coefficients relatively prime integers.

These results demonstrate that an even spin axisymmetric polarization tensor can be written as

$$U^{pol}_{ij}(\boldsymbol{k}) = U^{pol}(\boldsymbol{k}) S^{pol}_{ij}(\boldsymbol{k}) \tag{38}$$

where the scalar $U^{pol}(\boldsymbol{k})$ can be expanded in the series

$$U^{pol}(\boldsymbol{k}) = \sum_{\nu \geq 0 \text{ even}} B_{\nu}(k) k^{-\nu} Z^{\nu}(\boldsymbol{k}) \tag{39}$$

Eq. (38) with Eq. (39) is a tensor analog of the spherical harmonics expansion Eq. (18). Thus, for axial symmetry

$$
\begin{aligned}
H^{pol}_{ijmn} k^{-2} k_m k_n &= B_2(k) Z^2(\boldsymbol{k}) S^{pol}_{ij}(\boldsymbol{k}) \\
H^{pol}_{ijmnrs} k^{-4} k_m k_n k_r k_s &= B_4(k) Z^4(\boldsymbol{k}) S^{pol}_{ij}(\boldsymbol{k}) \qquad (40)
\end{aligned}
$$

The perhaps surprising conclusion that directional and polarization anisotropy are expanded in different basis functions is reflected in the orthogonality conditions. The orthogonality relation

$$\oint dS(\boldsymbol{k}) Y^\nu(\boldsymbol{k}) Y^\mu(\boldsymbol{k}) \propto \delta_{\nu\mu} \qquad (41)$$

applies equally to the entire contribution to the directional tensor $Y^\nu(\boldsymbol{k}) P_{ij}(\boldsymbol{k})$ because

$$Y^\nu(\boldsymbol{k}) P_{ij}(\boldsymbol{k}) Y^\mu(\boldsymbol{k}) P_{ij}(\boldsymbol{k}) = 2 Y^\nu(\boldsymbol{k}) Y^\mu(\boldsymbol{k}) \qquad (42)$$

Likewise, the appropriate orthogonality condition for polarization is the orthogonality of the tensors $Z^\nu(\boldsymbol{k}) S^{pol}_{ij}(\boldsymbol{k})$. Routine calculation shows that

$$\mathsf{S}^{pol} : \mathsf{S}^{pol} = \tfrac{1}{2} \left[k^2 - (\boldsymbol{a} \cdot \boldsymbol{k})^2 \right]^2 \qquad (43)$$

Orthogonality properties of associated Legendre polynomials [18] imply

$$
\begin{aligned}
&\oint dS(\boldsymbol{k}) Z^\nu(\boldsymbol{k}) S^{pol}_{mn}(\boldsymbol{k}) Z^\mu(\boldsymbol{k}) S^{pol}_{mn}(\boldsymbol{k}) \\
&= \tfrac{1}{2} \oint dS(\boldsymbol{k}) Z^\nu(\boldsymbol{k}) Z^\mu(\boldsymbol{k}) \left[k^2 - (\boldsymbol{a} \cdot \boldsymbol{k})^2 \right]^2 \propto \delta_{\mu\nu} \qquad (44)
\end{aligned}
$$

By analogy to the introduction of a basis of spherical harmonics in the analysis of directional anisotropy, we next construct a basis for the spaces of tensors defined by Eq. (30). Define

$$Y^{\nu,\mu}_{ij}(\boldsymbol{k}) + i Y^{\nu,-\mu}_{ij}(\boldsymbol{k}) = c_{\nu,\mu} \mathcal{L}^\nu_{ij} \left[Y^{\nu,\mu}(\boldsymbol{k}) \right] = c_{\nu,\mu} \mathcal{L}^\nu_{ij} \left[(k_x + i k_y)^\mu \frac{\partial^\mu}{\partial k_z^\mu} Y^\nu(\boldsymbol{k}) \right] \qquad (45)$$

It is shown in Appendix A that the constant $c_{\nu,\mu}$ can be chosen so that

$$Y^{\nu,\mu}_{ij}(\boldsymbol{k}) + i Y^{\nu,-\mu}_{ij}(\boldsymbol{k})$$

$$
\begin{aligned}
&= \mu(\mu-1)(k_x+ik_y)^{\mu-2}\left(\frac{\partial^{\mu}}{\partial k_z^{\mu}}Y^{\nu}(\boldsymbol{k})\right)\left(Y_{ij}^{2,2}(\boldsymbol{k})+iY_{ij}^{2,-2}(\boldsymbol{k})\right)\\
&+2\mu(k_x+ik_y)^{\mu-1}\left(\frac{\partial^{\mu+1}}{\partial k_z^{\mu+1}}Y^{\nu}(\boldsymbol{k})\right)\left(Y_{ij}^{2,1}(\boldsymbol{k})+iY_{ij}^{2,-1}(\boldsymbol{k})\right)\\
&+(k_x+ik_y)^{\mu}\left(\frac{\partial^{\mu+2}}{\partial k_z^{\mu+2}}Y^{\nu}(\boldsymbol{k})\right)Y_{ij}^{2,0}(\boldsymbol{k})
\end{aligned}
\tag{46}
$$

where

$$
\begin{aligned}
Y_{ij}^{2,2}(\boldsymbol{k})+iY_{ij}^{2,-2}(\boldsymbol{k}) &= \mathcal{L}_{ij}^{2}[(k_x+ik_y)^2]\\
Y_{ij}^{2,1}(\boldsymbol{k})+iY_{ij}^{2,-1}(\boldsymbol{k}) &= \mathcal{L}_{ij}^{2}[(k_x+ik_y)z]\\
Y_{ij}^{2,0}(\boldsymbol{k}) &= \mathcal{L}_{ij}^{2}[k^2-3k_z^2]
\end{aligned}
\tag{47}
$$

Appendix A includes explicit expressions for the tensors $Y_{ij}^{2,\mu}$. Note that

$$
Y_{ij}^{2,0}(\boldsymbol{k}) = S_{ij}^{pol}(\boldsymbol{k}) \tag{48}
$$

It is convenient to call the $Y_{ij}^{\nu,\mu}(\boldsymbol{k})$ *tensor spherical harmonics*, although they are subject to the special constraint of being polarization tensors. The five tensors $Y_{ij}^{2,\mu}(\boldsymbol{k})$ with $-2 \le \nu \le 2$ form a basis for even spin tensor spherical harmonics. This result generalizes the representation of axial symmetry by one anisotropic tensor to arbitrary anisotropy. A general even spin polarization tensor therefore has the expansion

$$
U_{ij}^{pol}(\boldsymbol{k}) = \sum_{\nu\ge 0 \text{ even}}\ \sum_{-\nu\le\mu\le\nu} A_{\nu,\mu}(k)Y_{ij}^{\nu,\mu}(\boldsymbol{k}) \tag{49}
$$

The use of a basis with five elements to describe even spin polarization tensors appears to be at variance with an elementary geometric argument [8] (see also Ref. [6]) that a basis of two tensors is sufficient. Given a polar axis $\boldsymbol{a}$, a local frame at any wavevector $\boldsymbol{k}$ is defined by the unit vector $\boldsymbol{k}/k$ and two orthogonal unit vectors in the plane perpendicular to $\boldsymbol{k}$ [19],

$$
\boldsymbol{e}^1(\boldsymbol{k}) = \boldsymbol{k}\times\boldsymbol{a}/|\boldsymbol{k}\times\boldsymbol{a}| \quad \boldsymbol{e}^2(\boldsymbol{k}) = \boldsymbol{k}\times\boldsymbol{e}^1/|\boldsymbol{k}\times\boldsymbol{e}^1| \tag{50}
$$

These are the usual unit vectors in spherical coordinates. In rectangular coordinates,

$$\boldsymbol{e}^1(\boldsymbol{k}) = (k_y, -k_x, 0)/\sqrt{k_x^2 + k_y^2}$$
$$\boldsymbol{e}^2(\boldsymbol{k}) = (k_x k_z, k_y k_z, -(k_x^2 + k_y^2))/k\sqrt{k_x^2 + k_y^2} \tag{51}$$

Since $\boldsymbol{k}\cdot\boldsymbol{e}^1(\boldsymbol{k}) = \boldsymbol{k}\cdot\boldsymbol{e}^2(\boldsymbol{k}) = 0$, a solenoidal tensor can be expressed as a linear combination of the four tensor products $\boldsymbol{e}^1(\boldsymbol{k})\boldsymbol{e}^1(\boldsymbol{k})$, $\boldsymbol{e}^2(\boldsymbol{k})\boldsymbol{e}^2(\boldsymbol{k})$, $\boldsymbol{e}^1(\boldsymbol{k})\boldsymbol{e}^2(\boldsymbol{k})$, and $\boldsymbol{e}^2(\boldsymbol{k})\boldsymbol{e}^1(\boldsymbol{k})$; in particular, a polarization tensor can be written as a linear combination of the two tensors $\boldsymbol{e}^1\boldsymbol{e}^1 - \boldsymbol{e}^2\boldsymbol{e}^2$ and $\boldsymbol{e}^1\boldsymbol{e}^2 + \boldsymbol{e}^2\boldsymbol{e}^1$.

Let us apply this representation to the even spin tensor spherical harmonics of Eq. (47). Using Eq. (51) and the explicit expressions of Eq. (A8),

$$\mathsf{Y}^{2,0}(\boldsymbol{k}) = -(k_x^2 + k_y^2)(\boldsymbol{e}^1\boldsymbol{e}^1 - \boldsymbol{e}^2\boldsymbol{e}^2)$$
$$\mathsf{Y}^{2,1}(\boldsymbol{k}) = 2k_x k_z(\boldsymbol{e}^1\boldsymbol{e}^1 - \boldsymbol{e}^2\boldsymbol{e}^2) - 2k_y k(\boldsymbol{e}^1\boldsymbol{e}^2 + \boldsymbol{e}^2\boldsymbol{e}^1)$$
$$\mathsf{Y}^{2,-1}(\boldsymbol{k}) = 2k_y k_z(\boldsymbol{e}^1\boldsymbol{e}^1 - \boldsymbol{e}^2\boldsymbol{e}^2) + 2k_x k(\boldsymbol{e}^1\boldsymbol{e}^2 + \boldsymbol{e}^2\boldsymbol{e}^1)$$
$$\mathsf{Y}^{2,2}(\boldsymbol{k}) = -(k_x^2 - k_y^2)(\boldsymbol{e}^1\boldsymbol{e}^1 - \boldsymbol{e}^2\boldsymbol{e}^2) + 2kk_z\frac{2k_x k_y}{k_x^2 + k_y^2}(\boldsymbol{e}^1\boldsymbol{e}^2 + \boldsymbol{e}^2\boldsymbol{e}^1)$$
$$\mathsf{Y}^{2,-2}(\boldsymbol{k}) = -2k_x k_y(\boldsymbol{e}^1\boldsymbol{e}^1 - \boldsymbol{e}^2\boldsymbol{e}^2) - 2kk_z\frac{k_x^2 - k_y^2}{k_x^2 + k_y^2}(\boldsymbol{e}^1\boldsymbol{e}^2 + \boldsymbol{e}^2\boldsymbol{e}^1) \tag{52}$$

The formulas for $\mathsf{Y}^{2,\pm 2}$ in Eq. (52) contain rational functions that are undefined along the line $k_x = k_y = 0$. Geometrically, this is the consequence of attempting to express the two-dimensional quantities $\mathsf{Y}^{2,\pm 2}$ in a spherical coordinate system. Clearing fractions in these expressions results in equations with polynomial coefficients relating $(k_x^2 + k_y^2)\mathsf{Y}^{2,\pm 2}$ to the other $\mathsf{Y}^{2,\mu}$, but not to expressions for the $\mathsf{Y}^{2,\pm 2}$ themselves.

That a nonsingular representation requires a basis with more elements than the apparent minimum imposed by linear independence is a common situation in algebra: a related issue underlies discussion of the linear dependence of elements in an 'integrity basis' for expressing the Reynolds stresses in terms of the mean velocity gradient; compare Canuto and Dubovikov [**?**].

3.5 *Polarization anisotropy: odd spins*

The definition of $\mathcal{L}^{\nu}_{ij}$ implies that it can only generate a tensor with even degree polynomial components if ν is even. An interesting feature of polarization is that there is a second way to generate tensors with even degree polynomial components by operating on spherical harmonics of odd degree.

Again referring to the list of operators in Ref. [3] Eq. (10), define the linear combination of operators B_6 and B_8,

$$\mathcal{O}^{\nu}_{ij} = \epsilon_{ibc}\left(k^2 k_b \partial_c \partial_j - (\nu-1) k_j k_b \partial_c\right) + \epsilon_{jbc}\left(k^2 k_b \partial_c \partial_i - (\nu-1) k_i k_b \partial_c\right) \quad (53)$$

In order that $\mathcal{O}^{\nu}_{ij}[\Phi_\nu(\boldsymbol{k})]$ with Φ_ν homogeneous of degree ν have even degree polynomial entries, ν must be odd. Like $\mathcal{L}^{\nu}_{ij}$, $\mathcal{O}^{\nu}_{ij}$ operating on any homogeneous harmonic polyomial produces a polarization tensor. In the special case of axial symmetry

$$\mathcal{O}^{\nu}_{ij}[Y^{\nu}(\boldsymbol{k})] = Z^{\nu}(\boldsymbol{k}) K^{pol}_{ij}(\boldsymbol{k}) \quad (54)$$

where

$$\mathsf{K}^{pol}(\boldsymbol{k}) = k^2\left[(\boldsymbol{k}\times\boldsymbol{a})\boldsymbol{a} + \boldsymbol{a}(\boldsymbol{k}\times\boldsymbol{a})\right] - (\boldsymbol{a}\cdot\boldsymbol{k})\left[(\boldsymbol{k}\times\boldsymbol{a})\boldsymbol{k} + \boldsymbol{k}(\boldsymbol{k}\times\boldsymbol{a})\right] \quad (55)$$

Provided that ν is odd, so that $Z^{\nu}(-\boldsymbol{k})\mathsf{K}^{pol}(-\boldsymbol{k}) = Z^{\nu}(\boldsymbol{k})\mathsf{K}^{pol}(\boldsymbol{k})$, these tensors satisfy the condition expressed in Eq. (1). But they were excluded in Refs. [1, 2] and later papers on axial symmetry [21] by an additional requirement noted in the Introduction of invariance under reflections in the plane perpendicular to the symmetry axis. This requirement was expressed by Sreenivasan and Narasimha [21] as the absence of a preferred direction of the symmetry axis.

The simplest odd spin polarization tensor is the axisymmetric spin three tensor $(\boldsymbol{a}\cdot\boldsymbol{k})\mathsf{K}^{pol}(\boldsymbol{k})$ defined by Eq. (55) for $\nu = 3$: it is easily verified that there is no solenoidal tensor of spin one. That the breaking of reflection symmetry requires adding a third tensor to the basis of two tensors for reflection invariant axial symmetry was noted by Cambon and Jacquin [13]; see also Ref. [9]. Its connection to breaking of reflectional symmetry suggests that odd spin axial symmetry inhabits a different category than its even spin counterpart, which might be called *semi*-axial symmetry following Ref. [13]. Among numerous papers treating this symmetry breaking, we note Refs. [22] and [23].

The general spin three polarization tensor proves to be related to the *stropholysis* tensor introduced by Kassinos et al. [9]. Given a harmonic cubic polynomial $a_{pqr}(k)k_p k_q k_r$ we can form the spin three polarization tensor

$$\tfrac{1}{6}\mathcal{O}^{3}_{ij}[a_{pqr}(k)k_p k_q k_r] = \epsilon_{ibc}(k^2 k_b a_{cjr}(k)k_r - k_j k_b a_{pqc}(k)k_p k_q) + (ij) \quad (56)$$

where (ij) denotes index symmetrization. Consider a contribution to the correlation tensor of the form

$$U^{pol}_{ij}(\boldsymbol{k}) = \tfrac{1}{6}k^{-3}\mathcal{O}^{3}_{ij}[a_{pqr}k_p k_q k_r] \quad (57)$$

Following the prescriptions of Ref. [9], form the angular moment

$$
\begin{aligned}
M_{ijpq}(k) &= \oint dS(\boldsymbol{k})\ U_{ij}^{pol}(\boldsymbol{k})k^{-2}k_pk_q \\
&= (\tfrac{1}{3}\epsilon_{irc}a_{cjr}(k)\delta_{pq} - \tfrac{1}{15}\epsilon_{ipc}a_{cjq}(k) - \tfrac{1}{15}\epsilon_{iqc}a_{cjp}(k)) + (ij) \\
&= -\tfrac{2}{15}(\epsilon_{ipc}a_{cjq}(k) + \epsilon_{iqc}a_{cjp}(k)) + (ij)
\end{aligned} \tag{58}
$$

A 'stropholysis spectrum' can be defined by

$$
\begin{aligned}
Q_{ijk}(k) &= \epsilon_{ipq}M_{jqpk}(k) + (ij) = \epsilon_{ipq}(\epsilon_{jpc}a_{cqk}(k) + \epsilon_{jkc}a_{cqp}(k)) + (ij) \\
&= -\tfrac{1}{15}(\delta_{ij}\delta_{qc} - \delta_{ic}\delta_{jq})a_{cqk}(k) + (ij) = \tfrac{2}{15}a_{ijk}(k)
\end{aligned} \tag{59}
$$

and is thus proportional to the original spin three tensor a_{ijk}.

In Ref. [9], it is noted that the traces of the tensor Q_{ijk} all vanish in homogeneous turbulence (the construction of Ref. [9] has the advantage that it also applies to inhomogeneous turbulence). This observation already makes explicit the connection between stropholysis and spin three. It is encouraging that a construction with no explicit connection to group theory leads naturally to the general spin three polarization tensor found by SO(3) methods.

We note that in the special case of axial symmetry, the symmetry breaking term introduced in Eq. (2.5) of Ref. [9] is, up to a change in notation, identical to K^{pol}.

Returning to the general development, a basis for odd spin tensors is defined, by analogy to Eq. (45),

$$
Y_{ij}^{\nu,\mu}(\boldsymbol{k}) + iY_{ij}^{\nu,-\mu}(\boldsymbol{k}) \equiv c_{\nu,\mu}\mathcal{O}_{ij}^{\nu}\left[(k_x+ik_y)^{\mu}\frac{\partial^{\mu}}{\partial k_z^{\mu}}Y^{\nu}(\boldsymbol{k})\right] \tag{60}
$$

It is shown in Appendix A that the constant $c_{\nu,\mu}$ can be chosen so that

$$
\begin{aligned}
&Y_{ij}^{\nu,\mu}(\boldsymbol{k}) + iY_{ij}^{\nu,-\mu}(\boldsymbol{k}) \\
&= \mu(\mu-1)(k_x+ik_y)^{\mu-2}\left(\frac{\partial^{\mu}}{\partial k_z^{\mu}}Y^{\nu}(\boldsymbol{k})\right)(X_{ij}^{2,2}(\boldsymbol{k}) + iX_{ij}^{2,-2}(\boldsymbol{k})) \\
&+\mu(k_x+ik_y)^{\mu-1}\left(\frac{\partial^{\mu+1}}{\partial k_z^{\mu+1}}Y^{\nu}(\boldsymbol{k})\right)(X_{ij}^{2,1}(\boldsymbol{k}) + iX_{ij}^{2,-1}(\boldsymbol{k})) \\
&+(k_x+ik_y)^{\mu}\left(\frac{\partial^{\mu+2}}{\partial k_z^{\mu+2}}Y^{\nu}(\boldsymbol{k})\right)X_{ij}^{2,0}(\boldsymbol{k})
\end{aligned} \tag{61}
$$

where

$$\begin{aligned}
&X_{ij}^{2,2}(\boldsymbol{k}) + iX_{ij}^{2,-2}(\boldsymbol{k}) = \mathcal{O}_{ij}^2[(k_x + ik_y)^2] \\
&X_{ij}^{2,1}(\boldsymbol{k}) + iX_{ij}^{2,-1}(\boldsymbol{k}) = \mathcal{O}_{ij}^2[(k_x + ik_y)z] \\
&X_{ij}^{2,0}(\boldsymbol{k}) = \mathcal{O}_{ij}^2[k^2 - 3k_z^2] \qquad (62)
\end{aligned}$$

Explicit expressions for these tensors appear in Appendix A. Whereas the even spin basis $\mathsf{Y}^{2,\mu}$ was formed by tensors with even degree polynomial entries, the $\mathsf{X}^{2,\mu}$ prove to have odd degree polynomial entries. It will be clear from Appendix A that they will multiply homogeneous functions of $\boldsymbol{k}$ of odd degree.

Given these results, we can write any polarization tensor as

$$U_{ij}^{pol}(\boldsymbol{k}) = \sum_{\nu \geq 0} \sum_{-\nu \leq \mu \leq \nu} A_{\nu,\mu}(k) Y_{ij}^{\nu,\mu}(\boldsymbol{k}) \qquad (63)$$

where it is understood that if ν is even, the $Y^{\nu,\mu}$ are defined by Eq. (46), and if ν is odd, the $Y^{\nu,\mu}$ are defined by Eq. (61) instead. This result generalizes Eq. (49) by removing the restriction to even spins.

4 Antisymmetric contributions to the correlation tensor

We briefly outline the general SO(3) description of these quantities, which describe helicity; further details appear in Ref. [8]. Denote the antisymmetric part of the correlation tensor by $H_{ij}(\boldsymbol{k})$. Recall from the discussion following Eq. (2) that $H_{ij}(\boldsymbol{k})$ is purely imaginary. Three operators in Ref. [3] Eq. (10) generate antisymmetric tensors. The operator B_4 acting on an arbitrary scalar $\Phi(\boldsymbol{k})$ gives

$$H_{ij}(\boldsymbol{k}) = \epsilon_{ijp} k_p \Phi(\boldsymbol{k}). \qquad (64)$$

This tensor is obviously solenoidal and can therefore contribute to the correlation tensor. The symmetry condition Eq. (2) forces $\Phi(-\boldsymbol{k}) = \Phi(\boldsymbol{k})$, thus $\Phi(\boldsymbol{k})$ is expanded in even degree spherical harmonics. The spin zero component is the helicity spectrum as conventionally defined, and the higher spins correspond to anisotropic helicity.

The operator B_3 is $k_i\partial_j - k_j\partial_i$. Its solenoidal projection can be constructed as $P_{im}P_{jn}(k_m\partial_n - k_n\partial_m)$, which vanishes identically. Therefore, this operator does not contribute to the correlation tensor.

Finally, we have the operator B_2, $\epsilon_{ijp}\partial_p$ with the solenoidal projection

$$P_{im}(\boldsymbol{k})P_{jn}(\boldsymbol{k})\epsilon_{mnp}\partial_p = \epsilon_{ijp}\partial_p - k^{-2}(k_i k_m \epsilon_{mjp} + k_j k_n \epsilon_{inp})\partial_p$$
$$= \epsilon_{ijp}\partial_p + k^{-2}(k_i \epsilon_{jmp} - k_j \epsilon_{imp})k_m \partial_p \tag{65}$$

Note that

$$\epsilon_{ijr}\epsilon_{r\ell n}k_\ell \epsilon_{nmp}k_m\partial_p = (\delta_{i\ell}\delta_{jn} - \delta_{in}\delta_{j\ell})k_\ell\epsilon_{nmp}k_m\partial_p$$
$$= k_i\epsilon_{jmp}k_m\partial_p - k_j\epsilon_{imp}k_m\partial_p \tag{66}$$

where the index r is contracted first. But contracting the index n first instead,

$$\epsilon_{ijr}\epsilon_{r\ell n}k_\ell\epsilon_{nmp}k_m\partial_p = \epsilon_{ijr}(\delta_{rm}\delta_{\ell p} - \delta_{rp}\delta_{\ell m})k_\ell k_m\partial_p = \epsilon_{ijr}k_p k_r\partial_p - \epsilon_{ijr}k^2\partial_r$$
$$= \nu\epsilon_{ijr}k_r - \epsilon_{ijr}k^2\partial_r \tag{67}$$

Therefore, the solenoidal projection Eq. (65) can be rewritten as

$$P_{im}(\boldsymbol{k})P_{jn}(\boldsymbol{k})\epsilon_{mnp}\partial_p = \epsilon_{ijp}\partial_p + \nu k^{-2}\epsilon_{ijr}k_r - \epsilon_{ijr}\partial_r = \nu k^{-2}\epsilon_{ijr}k_r \tag{68}$$

This operator differs trivially from Eq. (64), which therefore generates the most general anisotropic helicity.

The antisymmetric part of the correlation can therefore be expanded in spherical harmonics as

$$H_{ij}(\boldsymbol{k}) = \epsilon_{ijp}k_p \sum_{\nu\geq 0 \text{ even}} \sum_{-\nu\leq\mu\leq\nu} C_{\nu,\mu}(k)Y^{\nu,\mu}(\boldsymbol{k}) \tag{69}$$

in agreement with the analysis in Ref. [8]. The tensors in this expansion are trivially orthogonal, because

$$\epsilon_{ijp}k_pY^{\nu,\mu}(\boldsymbol{k})\epsilon_{ijq}k_qY^{\nu',\mu'}(\boldsymbol{k}) \propto k^2Y^{\nu,\mu}(\boldsymbol{k})Y^{\nu',\mu'}(\boldsymbol{k}) \tag{70}$$

5 Summary and Conclusions

The directional-polarization decomposition expresses the correlation tensor as the sum of two terms, each of which belongs to a rotation-invariant vector space. It is therefore a natural first step in the SO(3) decomposition, and is in fact prior to any particular treatment of the directional and polarization components.

We reviewed the spherical harmonics expansion for a scalar field, which proves to be appropriate for directional anisotropy, and showed how to extend that expansion to polarization tensors. For the space of polarization tensors of spin ν, we constructed a basis of $2\nu + 1$ tensor spherical harmonics $\mathsf{Y}^{\nu,\mu}$ with $-\nu \leq \mu \leq \nu$. For even spins, these tensor spherical harmonics are linear combinations with polynomial coefficients of the five tensors $\mathsf{Y}^{2,\mu}$ with $-2 \leq \mu \leq 2$ (Eq. (47)). Similarly, the odd spin harmonics are linear combinations of the five tensors $\mathsf{X}^{2,\mu}$ (Eq. (62)).

The main result of this paper is the explicit expressions for the tensor spherical harmonics in terms of basis tensors for even and odd spins, with coefficients depending on derivatives of the usual scalar spherical harmonics. This result permits a complete, explicit description of the anisotropy of the velocity correlation tensor and thus provides a foundation for mathematical modeling of anisotropic turbulence.

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Appendix A: Explicit expressions for basis tensors

Using the solenoidal projection operator [7] of Eq. (32), define for any constant tensor a_{ij},

$$\begin{aligned} a_{ij}^{pol}(\boldsymbol{k}) &= \tfrac{1}{2}k^2(P_{im}(\boldsymbol{k})P_{jn}(\boldsymbol{k}) + P_{jm}(\boldsymbol{k})P_{in}(\boldsymbol{k}) - P_{ij}(\boldsymbol{k})P_{mn}(\boldsymbol{k}))a_{mn} \\ &= k^2 a_{ij} - k_i k_p a_{pj} - k_j k_p a_{pi} + \tfrac{1}{2}(\delta_{ij} + k^{-2}k_i k_j)(\mathsf{a} : \boldsymbol{kk}) - \tfrac{1}{2}(\text{tr } \mathsf{a})P_{ij}(\boldsymbol{k}) \end{aligned} \tag{A1}$$

Then a_{ij}^{pol} is the unique polarization tensor (up to multiplication by a function of k) with coefficients that are linear in a_{ij}. It is easily verified that

$$S_{ij}^{pol} = (a_i a_j)^{pol} \tag{A2}$$

Now denote by $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}$ the unit vectors along x, y, z, so that

$$k_x = \boldsymbol{b} \cdot \boldsymbol{k} \quad k_y = \boldsymbol{c} \cdot \boldsymbol{k} \quad k_z = \boldsymbol{a} \cdot \boldsymbol{k} \tag{A3}$$

Define the tensor spherical harmonics $\mathsf{Y}^{\nu,\mu}$ by operating on the scalar harmonics with $\mathcal{L}^\nu$:

$$Y^{\nu,\mu}_{ij}(\boldsymbol{k}) = c_{\nu,\mu}\mathcal{L}^\nu_{ij} Y^{\nu,\mu}(\boldsymbol{k}) \tag{A4}$$

where, as before, the constants $c_{\nu.\mu}$ will be chosen to simplify the results. Substituting Eq. (20) for the $Y^{\nu,\mu}$,

$$Y^{\nu,\mu}_{ij}(\boldsymbol{k}) + iY^{\nu,-\mu}_{ij}(\boldsymbol{k}) = c_{\nu,\mu}\mathcal{L}^\nu_{ij}\left[(k_x+ik_y)^\mu \frac{\partial^\mu}{\partial k_z^\mu} Y^\nu(\boldsymbol{k})\right] \tag{A5}$$

Using Eqs. (A3) to evaluate the partial derivatives,

$$\begin{aligned}
&Y^{\nu,\mu}_{ij}(\boldsymbol{k}) + iY^{\nu,-\mu}_{ij}(\boldsymbol{k})\\
&= \mu(\mu-1)(k_x+ik_y)^{\mu-2}\left(\frac{\partial^\mu}{\partial z^\mu}Y^\nu(\boldsymbol{k})\right)[(b_ib_j-c_ic_j)^{pol} + i(b_ic_j+c_ib_j)^{pol}]\\
&+2\mu(k_x+ik_y)^{\mu-1}\left(\frac{\partial^{\mu+1}}{\partial k_z^{\mu+1}}Y^\nu(\boldsymbol{k})\right)[(a_ib_j+a_jb_i)^{pol} + i(a_ic_j+a_jc_i)^{pol}]\\
&+(k_x+ik_y)^{\mu}\left(\frac{\partial^{\mu+2}}{\partial z^{\mu+2}}Y^\nu(\boldsymbol{k})\right)(a_ia_j)^{pol}
\end{aligned} \tag{A6}$$

Operating on the five spin two spherical harmonics $Y^{2,\mu}$ (Eq. (21)) with $\mathcal{L}^2_{ij}$, we obtain the tensor harmonics

$$\begin{aligned}
&\mathcal{L}^2_{ij}[(k_x+ik_y)^2] = 2[(b_i+ic_i)(b_j+ic_j)]^{pol}\\
&= 2(b_ib_j-c_ic_j)^{pol} + 2i(b_ic_j+c_ib_j)^{pol}\\
&\mathcal{L}^2_{ij}[(k_x+ik_y)k_z] = [(b_i+ic_i)a_j + a_i(b_j+ic_j)]^{pol}\\
&= (a_ib_j+a_jb_i)^{pol} + i(a_ic_j+a_jc_i)^{pol}\\
&\mathcal{L}^2_{ij}[k^2-3k_z^2] = -6(a_ia_j)^{pol}
\end{aligned} \tag{A7}$$

Separating real and imaginary parts,

$$\begin{aligned}
&Y^{2,0}_{ij}(\boldsymbol{k}) = (a_ia_j)^{pol}\\
&Y^{2,1}_{ij}(\boldsymbol{k}) = (a_ib_j+a_jb_i)^{pol} \quad Y^{2,-1}_{ij}(\boldsymbol{k}) = (a_ic_j+a_jc_i)^{pol}\\
&Y^{2,2}_{ij}(\boldsymbol{k}) = (b_ib_j-c_ic_j)^{pol} \quad Y^{2,-2}_{ij}(\boldsymbol{k}) = (b_ic_j+c_ib_j)^{pol}
\end{aligned} \tag{A8}$$

This establishes Eq. (46) and gives the explicit formulas for tensor harmonics noted in Section 3.

For odd spins, we simply modify this argument appropriately. The tensor spherical harmonics $\mathsf{Y}^{\nu,\mu}$ with odd ν are defined by operating on the scalar harmonics with $\mathcal{O}^{\nu}$:

$$Y_{ij}^{\nu,\mu}(\boldsymbol{k}) = c_{\nu,\mu}\mathcal{O}_{ij}^{\nu}Y^{\nu,\mu}(\boldsymbol{k}) \tag{A9}$$

Substituting Eq. (20) for the $Y^{\nu,\mu}$,

$$Y_{ij}^{\nu,\mu}(\boldsymbol{k}) + iY_{ij}^{\nu,-\mu}(\boldsymbol{k}) = c_{\nu,\mu}\mathcal{O}_{ij}^{\nu}\left[(k_x+ik_y)^{\mu}\frac{\partial^{\mu}}{\partial k_z^{\mu}}Y^{\nu}(\boldsymbol{k})\right] \tag{A10}$$

Then as before,

$$\begin{aligned}
&Y_{ij}^{\nu,\mu}(\boldsymbol{k}) + iY_{ij}^{\nu,-\mu}(\boldsymbol{k}) \\
&= \mu(\mu-1)(k_x+ik_y)^{\mu-2}\left(\frac{\partial^{\mu}}{\partial z^{\mu}}Y^{\nu}(\boldsymbol{k})\right)\times \\
&[(\boldsymbol{k}\times(\boldsymbol{b}+i\boldsymbol{c}))_i(b_j+ic_j)+(b_i+ic_i)(\boldsymbol{k}\times(\boldsymbol{b}+i\boldsymbol{c}))_j]^{pol} \\
&+2\mu(k_x+ik_y)^{\mu-1}\left(\frac{\partial^{\mu+1}}{\partial k_z^{\mu+1}}Y^{\nu}(\boldsymbol{k})\right)[(\boldsymbol{k}\times\boldsymbol{a})_i(b_j+ic_j)+(b_i+ic_i)(\boldsymbol{k}\times\boldsymbol{a})_j]^{pol} \\
&+(k_x+ik_y)^{\mu}\left(\frac{\partial^{\mu+2}}{\partial z^{\mu+2}}Y^{\nu}(\boldsymbol{k})\right)[(\boldsymbol{k}\times\boldsymbol{a})_i a_j + a_i(\boldsymbol{k}\times\boldsymbol{a})_j]^{pol}
\end{aligned} \tag{A11}$$

Next note that

$$\begin{aligned}
&\mathcal{O}_{ij}^{2}[(k_x+ik_y)^2] = [(\boldsymbol{k}\times(\boldsymbol{b}+i\boldsymbol{c}))_i(b_j+ic_j)+(b_i+ic_i)(\boldsymbol{k}\times(\boldsymbol{b}+i\boldsymbol{c}))_j]^{pol} \\
&\mathcal{O}_{ij}^{2}[(k_x+ik_y)k_z] = [(\boldsymbol{k}\times\boldsymbol{a})_i(b_j+ic_j)+(b_i+ic_i)(\boldsymbol{k}\times\boldsymbol{a})_j]^{pol} \\
&\mathcal{O}_{ij}^{2}[k^2-3k_z^2] = [(\boldsymbol{k}\times\boldsymbol{a})_i a_j + a_i(\boldsymbol{k}\times\boldsymbol{a})_j]^{pol}
\end{aligned} \tag{A12}$$

Define new quantities

$$\begin{aligned}
X_{ij}^{2,2}(\boldsymbol{k}) + iX_{ij}^{2,-2}(\boldsymbol{k}) &= \mathcal{O}_{ij}^{2}[(k_x+ik_y)^2] \\
X_{ij}^{2,1}(\boldsymbol{k}) + iX_{ij}^{2,-1}(\boldsymbol{k}) &= \mathcal{O}_{ij}^{2}[(k_x+ik_y)k_z] \\
X_{ij}^{2,0}(\boldsymbol{k}) &= \mathcal{O}_{ij}^{2}[k^2-3k_z^2]
\end{aligned} \tag{A13}$$

This establishes Eq. (61). Evaluating the cross products and separating real and imaginary parts gives explicit expressions for the $\mathsf{X}^{2,\mu}$,

$$
\begin{aligned}
X_{ij}^{2,2} &= -k_y(a_i a_j)^{pol} + k_z(a_i c_j + c_i a_j)^{pol} + k_y(b_i b_j)^{pol} - k_x(b_i c_j + c_i b_j)^{pol} \\
X_{ij}^{2,-2} &= -k_x(a_i a_j)^{pol} - k_z(a_i b_j + c_i b_j)^{pol} - k_x(c_i c_j)^{pol} + k_y(b_i c_j + c_i b_j)^{pol} \\
X_{ij}^{2,1} &= -k_y(a_i b_j + b_i a_j)^{pol} - k_x(a_i c_j + c_i a_j)^{pol} + 2k_z(b_i c_j + c_i b_j)^{pol} \\
X_{ij}^{2,-1} &= k_x(a_i b_j + b_i a_j)^{pol} - k_y(a_i c_j + c_i a_j)^{pol} - k_z(b_i b_j - c_i c_j)^{pol} \\
X_{ij}^{2,0} &= -k_x(a_i c_j + c_i a_j)^{pol} + k_y(a_i b_j + b_i a_j)^{pol}
\end{aligned}
\tag{A14}
$$

Confirming the comments in Section 3, these tensors have odd degree polynomial entries, but, since ν is odd, the coefficients in Eq. (A11) are also of odd degree.